THE PLANE CONTACT PROBLEM IN THE ELECTRODYNAMICS OF CONTINUOUS MEDIA IN THE PRESENCE OF THE HALL EFFECT

Yu. P. Emets

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Solutions are given for the distributions of current density, field strength and electric field potential in the neighborhood of the point of contact of two conducting media with different scalar electrical conductivities σ and Hall constants R_H. Problems of this type are encountered in magnetohydrodynamic theory, and in semiconductor physics, for example, in investigating the fields in piecewise-inhomogeneous media or on the electrodes in magnetohydrodynamic channels and electrical engineering apparatus. If one of the two media has ideal properties $\sigma = \infty$, $R_H = 0$, then within the framework of the approximate theory (the induced magnetic field is neglected) the problem reduces to finding an analytic function in the region occupied by the second medium, and this can often be solved by carrying out a conformal mapping of the region onto a polygon [1, 2]. In other cases the electric field in each medium depends jointly on the physical properties and geometries of regions of the two media, and a solution must be found which is joined at the contact. The theory of singular integral equations [3, 4] is a convenient mathematical tool for solving such problems.

1. We shall give a solution to the problem, assuming that the electrical contact is between two electrically conducting bodies. We shall assume that the region of contact is small compared with the radius of curvature of both bodies, so that each can be conveniently represented as a half-plane with a common boundary on the segment ab (Fig. 1a, b).

We shall assume that the external magnetic field $H(0, 0, H_z)$ is everywhere uniform, but is not the same in the upper and lower half-planes in the general case, and is in a direction normal to the current vector $j(x, y)$ and the electric field $E(x, y)$; the magnetic field arising from the currents under consideration will be neglected. It turns out from the system of equations

$$\mathbf{j} = \sigma\mathbf{E} - \frac{\omega\tau}{H}\mathbf{j}\times\mathbf{H}, \qquad \operatorname{div}\mathbf{j} = 0,$$
$$\operatorname{rot}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{H} = 0, \qquad \omega\tau = R_H\sigma H \qquad (1.1)$$

and the assumptions made that the current field satisfies the equations div j = 0 and rot j = 0. Thus, the complex potential of the electric current F(z)

$$\frac{dF(z)}{dz} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = j_x(x, y) - ij_y(x, y),$$
$$F(z) = P(x, y) + iQ(x, y) \qquad (z = x + iy),$$
$$\frac{d\Gamma(z)}{dz} = j(z) \qquad (1.2)$$

may be introduced just as in [5].

Here P and Q are the potential and force functions of the current, respectively.

We shall designate the half-plane Im z > 0 by S^+, and the half-plane Im z < 0 by S^- and take for the positive direction on the real axis that which leaves the region S^+ from the left.

We shall solve the problem of the field current distribution for the current flowing through the contact. We have the following boundary conditions: the normal component of the current and the tangential

Fig. 1

component of the electric field are continuous over the region of contact, while on the remaining sections of the real axis the normal component of the current is equal to zero in each region. These conditions lead to the following boundary value problem (indices 1 and 2 refer to S^+ and S^-, respectively):

$$\operatorname{Im} j_1(x) = \operatorname{Im} j_2(x), \quad -l < x < l \quad \text{at} \quad y = 0,$$
$$\operatorname{Re}\left\{\frac{1 + i\omega_1\tau_1}{\sigma_1} j_1(x)\right\} = \operatorname{Re}\left\{\frac{1 + i\omega_2\tau_2}{\sigma_2} j_2(x)\right\},$$
$$-l < x < l \quad \text{at} \quad y = 0,$$
$$\operatorname{Im} j_1(x) = \operatorname{Im} j_2(x) = 0, \quad |x| > l \quad \text{at} \quad y = 0. \quad (1.3)$$

Here the complex form of Ohm's law (1.1) is employed:

$$j(z) = j_x(x, y) - ij_y(x, y) = \frac{-\sigma}{1 + i\omega\tau}\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right), \quad (1.4)$$

where U is the electrostatic potential.

We must also indicate the position of the current sources and sinks in S^+ and S^-. For the sake of simplifying the formulas obtained below, we shall assume that there is a source of finite intensity situated in S^+ at the point $z = \infty$, and a sink in S^- at the point $z = \infty$. We then obtain the condition at infinity

$$j_i(z) = \frac{C_{i1}}{z} + O(z^{-2}) \qquad (i = 1, 2) \text{ when } |z| \to \infty. \quad (1.5)$$

2. In order to solve the problem we introduce the two piecewise-holomorphic functions

$$\Psi_1(z) = \begin{cases} \Psi_1^+(z) & \text{when } z \in S^+, \ \Psi_1^+(z) = j_1(z), \\ \Psi_1^-(z) & \text{when } z \in S^-, \ \Psi_1^-(z) = \overline{j_1(\bar{z})}, \end{cases}$$
$$\Psi_2(z) = \begin{cases} \Psi_2^+(z) & \text{when } z \in S^+, \ \Psi_2^+(z) = \overline{j_2(\bar{z})}, \\ \Psi_2^-(z) & \text{when } z \in S^-, \ \Psi_2^-(z) = j_2(z). \end{cases} \quad (2.1)$$

We shall denote the normal current component at the contact by h(x) and assume that it satisfies a Hölder boundary condition.

On the basis of the first and last boundary conditions of (1.3) the functions $\Psi_i(z)$ $(i = 1, 2)$ may be expressed in terms of h(x) using Schwarz's integral

$$\Psi_1(z) = \frac{-1}{\pi}\int_{-l}^{l}\frac{h(x)\,dx}{x-z}, \qquad \Psi_2(z) = \frac{1}{\pi}\int_{-l}^{l}\frac{h(x)\,dx}{x-z}. \quad (2.2)$$

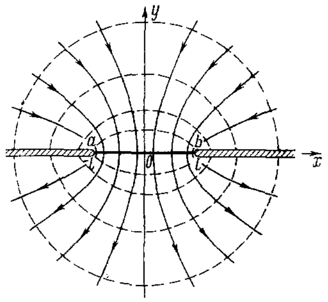

Fig. 2

Clearly, these functions satisfy condition (1.5) at infinity. Expanding the functions $\Psi_1(z)$ and $\Psi_2(z)$ at infinitely distant points in the regions S^+ and S^-, respectively, gives

$$\Psi_1(z) = $$

$$= \frac{z^{-1}}{\pi}\int_{-l}^{l} h(x)\,dx + \frac{z^{-2}}{\pi}\int_{-l}^{l} xh(x)\,dx + \ldots \text{ when } z \in S^+,$$

$$\Psi_2(z) =$$

$$= -\frac{z^{-1}}{\pi}\int_{-l}^{l} h(x)\,dx - \frac{z^{-2}}{\pi}\int_{-l}^{l} xh(x)\,dx - \ldots \text{ when } z \in S^-. \quad (2.3)$$

Whence we find the values of the constants C_{11} and C_{21} in (1.5):

$$C_{11} = \frac{1}{\pi}\int_{-l}^{l} h(x)\,dx, \quad C_{21} = -\frac{1}{\pi}\int_{-l}^{l} h(x)\,dx,$$

$$C_{11} = -C_{21} = \pi^{-1} I, \quad (2.4)$$

where I is the magnitude of the total current flowing through the contact which is a given quantity in the problem, or the intensity of the source situated at the point $z = \infty$ in the region S^+, which is the same thing. Taking (2.1) and (2.2) into account, we represent the second boundary condition of (1.3) in the form

$$\frac{1+i\omega_1\tau_1}{\sigma_1}\Psi_1^+(x) + \frac{1-i\omega_1\tau_1}{\sigma_1}\Psi_1^-(x) =$$

$$= \frac{1+i\omega_2\tau_2}{\sigma_2}\Psi_2^+(x) + \frac{1-i\omega_2\tau_2}{\sigma_2}\Psi_2^-(x)$$

$$-l < x < l \quad \text{at } y = 0. \quad (2.5)$$

Hence, making use of the Sokhotskii–Plemel formula, we obtain a homogeneous singular integral equation which must be satisfied by the function h(x):

$$(\sigma_1\omega_2\tau_2 - \sigma_2\omega_1\tau_1)\,h(x) + \frac{\sigma_1+\sigma_2}{\pi}\int_{-l}^{l}\frac{h(t)\,dt}{t-x} = 0$$

$$(-l < x,\ t < l). \quad (2.6)$$

This equation corresponds to the generalized linear boundary value problem (Riemann's problem) [3, 4]; for the function $\Psi_1(z)$ it has the form

$$\Psi_1^+(x) = -\frac{\sigma_1+\sigma_2+i(\sigma_1\omega_2\tau_2-\sigma_2\omega_1\tau_1)}{\sigma_1+\sigma_2-i(\sigma_1\omega_2\tau_2-\sigma_2\omega_1\tau_1)}\Psi_1^-(x)$$

$$-l < x < l \quad \text{at } y = 0,$$

$$\Psi_1^+(x) = \Psi_1^-(x) \qquad |x| > l \quad \text{at } y = 0. \quad (2.7)$$

Starting from the physical assumption that there is an accumulation of current on the ends of the contact, we find the solution of the boundary value problem in a class of functions having integrable singularities at the vertices *a* and b. By condition (1.5) this solution vanishes at infinity:

$$\Psi_1(z) = C_{11}(z+l)^{-1/2-\varepsilon}(z-l)^{-1/2+\varepsilon}$$

$$\varepsilon = \frac{1}{\pi}\operatorname{arc\,tg}\frac{\sigma_1\omega_2\tau_2-\sigma_2\omega_1\tau_1}{\sigma_1+\sigma_2}, \qquad -\frac{1}{2} < \varepsilon < \frac{1}{2}$$

$$\left((z+l)^{-1/2-\varepsilon}(z-l)^{-1/2+\varepsilon} = \frac{1}{z} + O(z^{-2}) \text{ when } |z| \to \infty\right). \quad (2.8)$$

According to (2.1) the function $\Psi_1(z)$ should satisfy the condition

$$\Psi_1(z) = \overline{\Psi_1(\bar{z})},$$

from which it follows that the constant C_{11} should be a real quantity. From (2.8) we find all the required fields in S^+.

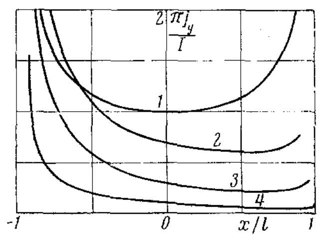

Fig. 3

The current distribution in S^+ is determined directly by the function $\Psi_1(z)$

$$j_1(z) = j_{1x}(x, y) - ij_{1y}(x, y) =$$

$$= C_{11}(z+l)^{1/2-\varepsilon}(z-l)^{-1/2+\varepsilon} \text{ when } z \in S^+, \quad (2.9)$$

and is found from the Sokhotskii–Plemel formulas on the contact *ab*

$$j_{1y}(x, 0) = h(x) = \frac{\Psi_1^-(x) - \Psi_1^+(x)}{2i} =$$

$$= C_{11}(l+x)^{-1/2-\varepsilon}(l-x)^{-1/2+\varepsilon} \times$$

$$\times \left[1 + \left(\frac{\sigma_1\omega_2\tau_2-\sigma_2\omega_1\tau_1}{\sigma_1+\sigma_2}\right)^2\right]^{-1/2} \qquad (-l < x < l)$$

$$j_{1x}(x, 0) = \frac{\Psi_1^+(x) + \Psi_1^-(x)}{2} =$$

$$= \frac{1}{\sigma_1+\sigma_2}C_{11}(\sigma_1\omega_2\tau_2-\sigma_2\omega_1\tau_1)(l+x)^{-1/2-\varepsilon} \times$$

$$\times (l-x)^{-1/2+\varepsilon}\left[1 + \left(\frac{\sigma_1\omega_2\tau_2-\sigma_2\omega_1\tau_1}{\sigma_1+\sigma_2}\right)^2\right]^{-1/2} (-l < x < l). \quad (2.10)$$

The constant C_{11} is determined from the condition

$$\int_{-l}^{l} j_{1y}(x, 0)\,dx = I. \qquad (2.11)$$

Using the familiar formula

$$\int_{-1}^{1} (1+t)^{-1/2-\varepsilon}(1-t)^{-1/2+\varepsilon}\,dt =$$

$$= \Gamma\left(\frac{1}{2}-\varepsilon\right)\left(\frac{1}{2}+\varepsilon\right) = \frac{\pi}{\cos \pi\varepsilon} \qquad (2.12)$$

we may easily verify that in fact

$$C_{11} = \pi^{-1} I. \qquad (2.13)$$

Further, we find the electric field in S^+

$$E_1(z) = -\frac{\partial U_1}{\partial x} + i\frac{\partial U_1}{\partial y} =$$

$$= (1 + i\omega_1\tau_1)\frac{I}{\pi\sigma_1}(z+l)^{-1/2-\varepsilon}(z-l)^{-1/2+\varepsilon} \text{ when } z \in S^+, \qquad (2.14)$$

$$U_1(x, y) = U_1(x_0, y_0) +$$

$$+ \mathrm{Re}\left\{-(1+i\omega_1\tau_1)\frac{I}{\pi\sigma_1}\int_{z_0}^{z}(z+l)^{-1/2-\varepsilon}(z-l)^{-1/2+\varepsilon}\,dz\right\}$$

$$\text{when } z \in S^-. \qquad (2.15)$$

In formula (2.15) the integral is expressible in terms of elementary functions only when the exponent $1/2 + \varepsilon$ is a rational number. In this case we set $1/2 + \varepsilon = p/q$ (p, q are natural numbers; $p < q$) and in (2.15) we make the substitution

$$\left(\frac{z-l}{z+l}\right)^{1/q} = t. \qquad (2.16)$$

This substution reduces (2.15) to the form

$$U_1(t) = U_1(t_0) + \mathrm{Re}\left\{-(1+i\omega_1\tau_1)\frac{I}{\pi\sigma_1}\int_{t_0}^{t}\frac{t^{p-1}dt}{1-t^q}\right\}. \qquad (2.17)$$

Now expanding the integrand in simple fractions, instead of (2.15) we have

$$U_1(x, y) = U_1(x_0, y_0) +$$

$$+ \mathrm{Re}\left\{-(1+i\omega_1\tau_1)\frac{I}{\pi\sigma_1}\frac{1}{q}\sum_{\nu=0}^{q-1}\exp\left[\frac{2\pi i\nu(p-q)}{q}\right] \times \right. \qquad (2.18)$$

$$\left. \times \ln\left[\exp\left(\frac{2\pi i\nu}{q}\right) - \left(\frac{z-l}{z+l}\right)^{1/q}\right]\left[\exp\left(\frac{2\pi i\nu}{q}\right) - \left(\frac{z_0-l}{z_0+l}\right)^{1/q}\right]^{-1}\right\}.$$

Formulas similar to those already given may be obtained for the region S^- also, while it turns out that at the contact the relation

$$j_{1x}(x, 0) = -j_{2x}(x, 0), \qquad -l < x < l \qquad (2.19)$$

holds, which is a consequence of the complete symmetry of the regions S^+ and S^- in the gemetrical respect and in the way the boundary conditions are specified.

We shall analyze the expressions for the current (2.9), (2.10) in the following cases.

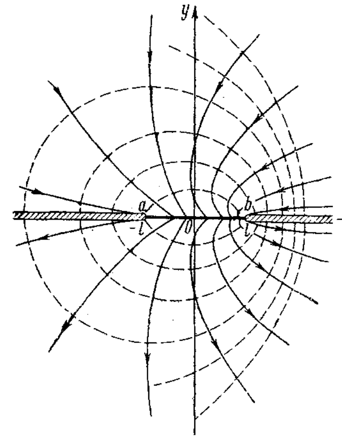

Fig. 4

(a) Let there be no Hall effect in the two media ($\omega_1\tau_1 = \omega_2\tau_2 = 0$), and let each of them have finite conductivities σ_1 and σ_2. It then follows from (2.9)–(2.10) that

$$j_1(z) = \frac{I}{\pi\sqrt{z^2-l^2}} \quad \text{when } z \in S^+, \qquad (2.20)$$

$$j_{1y}(x, 0) = \frac{I}{\pi\sqrt{l^2-x^2}}, \quad j_{1x}(x, 0) = 0,$$

$$-l < x < l. \qquad (2.21)$$

Figure 2 gives a qualitative picture of the lines of force (continuous curves) and potential lines (broken curves) of the electric current described by expression (2.20). The absence of a tangential current component on the contact is explained by the symmetry of the current distribution in the two media relative to the ordinate axis. The function (2.21) is illustrated by curve 1 in Fig. 3.

(b) We shall consider the case when the Hall effect appears only in the medium occupying the region S^+, ($\omega_1\tau_1 \neq 0$, $\omega_2\tau_2 = 0$), and the conductivity of the two media is finite as before. The quantities $j_1(z)$, $j_{1y}(x, 0)$ and $j_{1x}(x, 0)$ in S are now determined by the formulas

$$j_1(z) = \frac{I}{\pi}(z+l)^{-1/2-\varepsilon_1}(z-l)^{-1/2+\varepsilon_1} \quad \text{when } z \in S^+$$

$$j_{1y}(x, 0) = \frac{1}{\pi} I (l+x)^{-1/2-\varepsilon_1}(l-x)^{-1/2+\varepsilon_1}\left(1 + \frac{\sigma_2^2\omega_1^2\tau_1^2}{(\sigma_1+\sigma_2)^2}\right)^{-1/2},$$

$$-l < x < l,$$

$$j_{1x}(x, 0) = \frac{-\sigma_2\omega_1\tau_1 I}{\pi(\sigma_1+\sigma_2)}(l+x)^{-1/2-\varepsilon_1}(l-x)^{-1/2+\varepsilon_1}\left(1 + \frac{\sigma_2^2\omega_1^2\tau_1^2}{(\sigma_1+\sigma_2)^2}\right)^{-1/2},$$

$$-l < x < l$$

$$\left(\varepsilon_1 = \frac{1}{\pi}\,\mathrm{arc\,tg}\,\frac{-\sigma_2\omega_1\tau_1}{\sigma_1+\sigma_2},\ 0 \leqslant |\varepsilon| < \frac{1}{2}\right). \qquad (2.22)$$

A qualitative picture of the current distribution for the region S^-, obtained from (2.22) and similar formulas, is given in Fig. 4. The influence of the Hall effect is apparent in the bending of the current lines in the segment ab and in an increase of their density along one of the ends of the contact. The current concentration along this end increases as the parameter $\omega_1\tau_1$ increases.

Figure 3 gives curves 2, 3 and 4, which characterize the distribution of normal current density in the segment ab for the three values of the dimensionless parameter $\sigma_2\omega_1\tau_1/(\sigma_1+\sigma_2) = 1$, 3 and 10, respectively (current concentration along end *a* of the contact). For a fixed value of the parameter $\omega_1\tau_1$ the current concentration on the contact may be weakened by decreasing the conductivity of the medium occupying the region S^-.

(c) In the general case, when the Hall effect exists in the two media, the qualitative picture of the current distribution is the same as that given in Fig. 4, and all the points discussed in (b) remain valid. They must, however, be supplemented as follows.

If the equation

$$\frac{\omega_1\tau_1}{\sigma_1} = \frac{\omega_2\tau_2}{\sigma_2} \tag{2.23}$$

holds, which, taking (1.1) into account, may also be represented in the form

$$R_{1H} H_1 = R_{2H} H_2, \tag{2.24}$$

then from expressions (2.9), (2.10) we arrive at the formulas (2.20), (2.21), describing the current distribution when the Hall effect is absent from both media. This is explained by the fact that, observing condition (2.24), the Hall phenomenon manifests itself equally in the two media, and in this case exerts no influence on the current distribution (the two media behave like a single conductor in which cuts are made along the rays $(-\infty, -1)$, $(1, \infty)$).

Condition (2.24) may be considered as the limiting case of the two inequalities

$$R_{1H} H_1 > R_{2H}H_2, \qquad R_{1H} H_1 < R_{2H} H_2, \tag{2.25}$$

each of which indicates along which of the two ends of the contact the current concentration arises.

A charge layer is formed at the contact of the two media, which is determined from the condition

$$E_{2y} - E_{1y} = 4\pi\rho_e, \quad -l < x < l \quad \text{at} \quad y = 0, \tag{2.26}$$

or, if we make use of Ohm's law, from the second condition

$$\frac{1}{\sigma_1} j_{2y}(x, 0) - \frac{\omega_2\tau_2}{\sigma_2} j_{2x}(x, 0) - \frac{1}{\sigma_1} j_{1y}(x, 0) + \frac{\omega_1\tau_1}{\sigma_1} j_{1x}(x, 0) = 4\pi\rho_e,$$

$$-l < x < l. \tag{2.27}$$

Hence we find

$$\rho_e = \frac{1}{4\pi^2\sigma_1\sigma_2(\sigma_1 + \sigma_2)} I\,[\sigma_1^2(1 + \omega_2^2\tau_2^2) - \sigma_2^2(1 + \omega_1^2\tau_1^2)] \times$$

$$\times (l + x)^{-1/2-\varepsilon}(l - x)^{-1/2+\varepsilon} \times$$

$$\times \left[1 + \left(\frac{\sigma_1\omega_2\tau_2 - \sigma_2\omega_1\tau_1}{\sigma_1 + \sigma_2}\right)^2\right]^{1/2} \quad -l < x < l \quad \text{at} \quad y = 0 \tag{2.28}$$

Where the linear charge distribution on the segment ab is found all the basic characteristics of the field in the neighborhood of the contact are determined.

The problem considered may easily be generalized to include the case when contact between electrically conducting bodies occurs over several segments.

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